HYPERSTATIONARY SETS

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For κ a regular uncountable cardinal, the unbounded subsets of κ are the positive sets with respect to the Fréchet filter on κ (i.e., the set of subsets of κ whose complement has cardinality less than κ), whereas the stationary sets are the positive sets with respect to the closed unbounded (club) filter on κ. There is a potential hierarchy of filters, extending the club filter, whose positive sets give rise to the notion of hyperstationary set (i.e. ξ-stationary for some $\xi > 1$): We say that a subset A of some limit ordinal κ is *0-stationary* if it is unbounded. For $\xi > 0$, we say that A is *ξ-stationary* if and only if for every $\zeta < \xi$, every pair of subsets S and T of κ that are ζ-stationary *simultaneously ζ-reflect* to some $\alpha \in A$, i.e., $S \cap \alpha$ and $T \cap \alpha$ are both ζ-stationary in α.

Thus, $A \subseteq \kappa$ is 0-stationary iff it is unbounded, it is 1-stationary iff it is stationary, and it is 2-stationary iff every stationary $S \subseteq \kappa$ reflects to some $\alpha \in A$, i.e., $S \cap \alpha$ is stationary in α. Writing $\mathcal{F}^\xi_\kappa$ for the set $\{X \subseteq \kappa : \kappa - X \text{ is not } \xi\text{-stationary}\}$, we have that $\mathcal{F}^0_\kappa$ is the Fréchet filter on κ, and $\mathcal{F}^1_\kappa$ is the club filter. In general, $\mathcal{F}^\xi_\kappa$, for $\xi \geq 2$, is a filter iff κ is ξ-stationary ([1, 3]).

Now it turns out that for the filters $\mathcal{F}^\xi_\kappa$, $\xi \geq 2$, to be non-trivial, large cardinals are needed. Indeed, the existence of a 2-stationary cardinal κ is equiconsistent with the existence of a weakly compact cardinal ([10]). Moreover, in the constructible universe, L, a regular cardinal κ is $(\xi+1)$-stationary iff it is Π^1_ξ-indescribable ([2,3]).

The original motivation for the introduction and study of ξ-stationary sets was the still open problem of the ordinal topological completeness of Generalized Provability Logics $\mathbf{GLP}_\xi$, for $\xi > 2$ ([4,8,9], [5,6]). The ordinal topologies $\langle \tau_\zeta : \zeta < \xi \rangle$ involved in any proof of completeness of $\mathbf{GLP}_\xi$ must be non-discrete, and the non-isolated points of the τ_ζ topology are exactly the ordinals that are ζ-stationary ([2]).

We shall discuss the intriguing connections between hyperstationary sets, large cardinals, the normality of the $\mathcal{F}^\xi_\kappa$ filters, their corresponding ordinal topologies, and the key combinatorial issues involved in the (possible) proof of ordinal topological completeness of $\mathbf{GLP}_\xi$. New interesting set-theoretical notions, such as *hypercofinalities* or *hypersquares* ([7]) come naturally out of these connections.

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