SWITCHING LEMMAS AND PROPOSITIONAL PROOF COMPLEXITY

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Switching lemmas are basic tools in computational complexity theory. The original switching lemma, due to Furst, Saxe and Sipser [5] and Ajtai [2] states that we can switch efficiently between conjunctive normal form (CNF) and disjunctive normal forms (DNF) of a Boolean function, provided that we first subject the function to a random restriction. Using their switching lemma, they were able to demonstrate super-polynomial lower bounds on the size of bounded-depth circuits computing the parity function.

Ajtai in a groundbreaking paper of 1988 [1] extended the scope of switching lemmas beyond the framework of classical Boolean logic used in the circuit lower bounds. Working in a nonstandard model of arithmetic, he showed that the pigeonhole principle, expressed as a tautology in classical logic has no polynomial-size proofs in a Frege system for classical logic in which all the formulas in the proofs are restricted to a fixed depth. Ajtai's proof uses a modified version of the switching lemma that works in this nonstandard context, where the partial assignments (restrictions) used in the classical case are replaced by partial matchings in the complete bipartite graph $K_{n+1,n}$.

Subsequently, two groups of researchers, Paul Beame, Russell Impagliazzo, Toniann Pitassi and Jan Krajíček, Pavel Pudlák, Alan Woods [4, 7, 6] independently extended Ajtai's result in two ways, first by providing a standard proof of the lower bound, dispensing with nonstandard models, second by raising the lower bound to exponential.

The proofs of the switching lemmas described above all used intricate probabilistic reasoning. Later, Razborov found a much simpler argument that employed the idea of mapping the set of "bad" restrictions (the partial assignments that do not permit efficient switching) into a small set.

Paul Beame, in a technical report from 1994 [3] used Razborov's approach to prove not only the original switching lemma but also switching lemmas based on spaces of matchings, as used in complexity results for bounded depth Frege systems.

The proofs of switching lemmas in Beame's report, as well as in Urquhart and Fu's paper of 1996 [8] follow Razborov's basic strategy, though the details are in each case slightly different. This suggests that there might be an abstract version of a switching lemma, from which we can derive all of the particular cases by specialization. Providing such an abstract analysis is the main contribution of the present paper. In addition, we provide a general version of the lower bounds for bounded depth Frege systems.

References

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